SAMUEL G. HANSON
Harvard Business School

ADI SUNDERAM
Harvard Business School

JEREMY C. STEIN
Harvard University

ERIC ZWICK
University of Chicago Booth School of Business

Appendix For:
Business Credit Programs in the Pandemic Era

A. The Rationale for Credit-Market Intervention in a Pandemic

In this Appendix, we exposit the model in greater detail, explicitly filling in the parametric assumptions detailed in the text. We then solve the model by backwards inducting from $t = \infty$. Using these solutions, the formal propositions stated in the text all then follow directly.

A.1. Model Setting

Our model has three periods—which we label $t = 1$, $t = 2$, and $t = \infty$—and a continuum of firms $f \in [0,1]$ who differ solely in terms of their exposure to a negative real shock that first hits the economy at $t = 1$. All agents in the economy are risk neutral with a constant time-discount factor given by $\delta \in (0,1)$. We use F_{S_t} to denote the mass of firms that are operating in state S_t at time t. Specifically, it will turn out that all firms $f \in [0, F_{S_t}]$ will operate in state S_t at time t.

Each firm can be shut down at any date t. If a firm is shut down it generates zero free cash flow in the period it is first shut down and in all future periods. If a firm is operating in a given period, it generates some free cash flow. If this free cash flow is positive, some portion of it can be paid out to the firm's outside investors. If the free cash flow is negative, it represents a take-it-or-leave-it investment that outside investors must make if the firm is to stay alive.

If firm f is operating at $t = 1$ it generates a free cashflow of

$$X_1(f, R_1) = \mu + \gamma - R_1 - \Delta \times f \ . \tag{A1}$$

Here $\mu > 0, \gamma \geq 0$, and $R_1 > 0$ parameterizes the impact of the recession on firm's cashflow at $t = 1$.[1] We assume that $R_1 + \Delta > \mu + \gamma$, so $X_1(1, R_1) < 0$—i.e., the most exposed firms have negative free cashflow $t = 1$ and require outside investment if they are to survive to $t = 2$.

[1] All of our results remain qualitatively unchanged if we assume that the exposure of firm $f \in [0,1]$ to the aggregate recession state is given by an increasing linear function of f, $\beta[f]$. Thus, for instance, we can consider the case where firms with f near 0 have $\beta(f) < 0$. Firms with $\beta(f) < 0$ are negatively exposed to the recession shock and, therefore,

We assume all uncertainty is resolved at $t = 2$. There are two possible states at $t = 2$: $S_2 \in \{B_2, G_2\}$. With probability p the recession will be bad at $t = 2$, denoted as $S_2 = B_2$, and with probability $1 - p$ the recession will not be as bad, denoted as $S_2 = G_2$. If firm f is operating in state S_2 at $t = 2$ it generates a free cashflow of:

$$X_2(f, R_{S_2}, F_{S_2}) = \mu + \gamma \times F_{S_2} - R_{S_2} - \Delta \times f \ . \tag{A2}$$

We assume $R_{B_2} > R_{G_2} > 0$, which captures the idea that the recession is more severe in the bad state $t = 2$. In equation (A2), $\gamma \times F_{S_2} \geq 0$ is a reduced-form for the aggregate demand externality that we assume exists at $t = 2$: the cashflows of any individual firm are greater when more firms are operating at $t = 2$. We assume that $R_{G_2} + \Delta > \mu + \gamma$, so the most exposed firms require financing from outside investors in order to survive in the good state at $t = 2$ even if all firms in the economy are operating. We also assume that $\Delta - 2\gamma > 0$, implying that the aggregate demand externalities at $t = 2$ are not so powerful that the social planner will want to keep all firms alive at $t = 2$. Since it will turn out that more firms will continue operating when $S_2 = G_2$ than when $S_2 = B_2$—i.e., we will have $F_{G_2} > F_{B_2}$, it will be the case that $X_2(f, R_{G_2}, F_{G_2}) > X_2(f, R_{B_2}, F_{B_2})$: all firms will generate higher cashflows in the good state at $t = 2$ than in the bad state.

There are also two potential steady states at $t = \infty$: $S_\infty \in \{B_\infty, G_\infty\}$. Since all uncertainty is resolved at $t = 2$, we have $S_\infty = B_\infty$ if $S_2 = B_2$; similarly, we have $S_\infty = G_\infty$ if $S_2 = G_2$. If firm f is operating in state S_∞ at $t = \infty$, we assume it generates a free cashflow of

$$X_\infty(f, R_{S_\infty}) = \mu + \gamma - R_{S_\infty} - \Delta \times f \ , \tag{A3}$$

where $R_{S_2} > R_{S_\infty} > 0$ for $S \in \{B, G\}$ and $R_{B_\infty} > R_{G_\infty}$. Thus, all firms will generate greater cashflows at $t = \infty$ than at $t = 2$ regardless of the aggregate state—i.e., for $S \in \{B, G\}$, we have $X_\infty(f, R_{S_\infty}) > X_2(f, R_{S_2}, F_{S_2})$ for all f. And, all firms will generate more cashflows in the new steady-state if the recession is less severe—i.e. $X_\infty(f, R_{G_\infty}) > X_\infty(f, R_{B_\infty})$.

To introduce capital market frictions at $t = 1$, we assume that if $X_1(f, R_1) < 0$—i.e., if firm f requires an outside cash investment at $t = 1$ in order to survive to $t = 2$—that private investors can only appropriate a fraction $0 < (1 - \varphi) \leq 1$ of the firm's total value of $t = 2$ where $\varphi \in [0,1)$. Thus, capital markets are frictionless in the limit where $\varphi = 0$. Limited pledgeability constraints of this sort often emerge from moral hazard problems between investors and firm

will be free cashflow positive at $t = 1$, $t = 2$, and $t = \infty$. Trivially, such firms will always continue to operate. Thus, we assume for simplicity that all firms are positively exposed to the recession shock.

managers (Holmstrom and Tirole [1997]). Alternately, such a limited pledgeability constraint can be seen as a reduced form for the idea that some of the social surplus that a firm generates accrues to stakeholders other than the firm's investors and managers (e.g., to firm employees) or to other agents in the economy.

As equation (A3) shows, we assume there are no market failures in the new steady-state that begins at $t = \infty$. Thus, firms that are not viable in state S_∞—i.e., firms that have negative free cashflows in the long run—will be shut down by $t = \infty$ in state S_∞ in both the planner's solution and the private market solution. The viable firms at $t = \infty$ in state S_∞ satisfy $X_\infty(f, R_{S_\infty}) \geq 0$ or

$$f \leq \bar{F}_{S_\infty} = (\mu + \gamma - R_{S_\infty}) / \Delta . \quad \text{(A4)}$$

We assume that $R_{G_\infty} + \Delta > \mu + \gamma$, implying that $\bar{F}_{G_\infty} < 1$: some firms will not be viable in the long-run at $t = \infty$ even if the recession turns to be less severe. Of course, many firms that would be long-run viable in state S_∞ will require substantial investments in state S_2 at $t = 2$ and at $t = 1$ if they are to survive. If these investments are too large, they will not be worthwhile even though the firm would be viable in the long run. Thus, more firms that are long-run viable will survive to $t = \infty$ in the social planner's solution than in the private market outcome.

We solve the model by backwards inducting from $t = \infty$. Specifically, for each state S_t, we conjecture that we enter the state with all firms $f \in [0, F_{S_{t-1}}]$ still intact from the preceding state S_{t-1} at time $t - 1$. We then look for a new cutoff $F_{S_t} \leq F_{S_{t-1}}$ such that all firms $f \in [0, F_{S_t}]$ will continue operating in state S_t at time t. Thus, an equilibrium solution of our model is a set of five cutoffs $\{F_1, F_{G_2}, F_{B_2}, F_{G_\infty}, F_{B_\infty}\}$ that give the identity of the most-exposed firm that is still operating in each state. Naturally, the planner's solution and the private market solution will diverge when there are either aggregate demand externalities at $t = 2$ ($\gamma > 0$) or credit market frictions at $t = 1$ ($\varphi > 0$). We first explain the private market solution and then we explain how the planner's solution differs from the private market outcome. Here we are most interested in assessing how changes in model parameters impact the difference between these two solutions.

A.2. The Private Market Outcome

THE STEADY STATE We start in state S_∞ at $t = \infty$. Assuming they have survived at both $t = 1$ and $t = 2$, only viable firms with $X_\infty(f, R_{S_\infty}) \geq 0$ or firms that satisfy $f \leq \bar{F}_{S_\infty} = (\mu + \gamma - R_{S_\infty})/\Delta$ will continue operating in state S_∞ at $t = \infty$. Thus, if firm f survives until $t = \infty$ in state S_∞, its value to private investors will be

$$V_\infty(f,S_\infty)=\frac{1}{1-\delta}\cdot\min\{X_\infty(f,R_{S_\infty}),0\}=\frac{1}{1-\delta}\cdot[\mu+\gamma-R_{S_\infty}-\Delta\times f]\cdot 1_{\{f\le\bar{F}_{S_\infty}\}}, \tag{A5}$$

where $1_{\{f\le\bar{F}_{S_\infty}\}}$ is a binary indicator that switches on when $f\le\bar{F}_{S_\infty}$.

THE INTERIM DATE We now work backward to state S_2 at $t=2$. We suppose all firms $f\in[0,F_1]$ survived at $t=1$. If the mass of firms that continue operating in state S_2 at $t=2$ is equal to F_{S_2}, the private value of firm f:

$$\begin{aligned}V_2(f,R_{S_2},F_{S_2})&=\min\left\{X_2(f,R_{S_2},F_{S_2})+\delta\cdot V_\infty(f,R_{S_\infty}),0\right\}\\&=\min\left\{[\mu+\gamma\times F_{S_2}-R_{S_2}-\Delta\times f]+\frac{\delta}{1-\delta}\cdot[\mu+\gamma-R_{S_\infty}-\Delta\times f]\cdot 1_{\{f\le\bar{F}_{S_\infty}\}},0\right\}.\end{aligned} \tag{A6}$$

Inspecting equation (A6), we see that, holding fixed the number of firms that continue operating (F_{S_2}) in S_2, private firm value $V_2(f,R_{S_2},F_{S_2})$ is (weakly) decreasing in a given firm's exposure f. At the same time, the private value of any firm f is (weakly) increasing in the mass of firms still operating, F_{S_2}, which reflect the aggregate demand externalities. However, since $\Delta-\gamma>0$, the value of the marginal firm that continues operating in S_2, namely $V_2(F_{S_2},R_{S_2},F_{S_2})$, is still (weakly) decreasing in the exposure of the marginal firm, F_{S_2}.

There are two cases to consider at S_2. First, if all firms that survive at $t=1$ are still privately valuable in state S_2 at $t=2$, then no additional firms will be shut down in S_2. Formally, if $V_2(F_1,R_{S_2},F_1)\ge 0$, we must then have $F^*_{S_2}=F_1$. By contrast, if $V_2(F_1,R_{S_2},F_1)<0$, we must have $F^*_{S_2}=\hat{F}^*_{S_2}<F_1$ where $\hat{F}^*_{S_2}$ is the solution to $V_2(\hat{F}^*_{S_2},R_{S_2},\hat{F}^*_{S_2})=0$. Given our assumptions, this marginal firm requires outside investment in state S_2 (i.e., $X_2(\hat{F}^*_{S_2},R_{S_2},\hat{F}^*_{S_2})<0$), but has positive value in state S_∞ at $t=\infty$ (i.e., $V_\infty(\hat{F}^*_{S_2},R_{S_2})>0$). Thus, $\hat{F}^*_{S_2}$ is given by:

$$\hat{F}^*_{S_2}=\frac{(1-\delta)\cdot(\mu-R_{S_2})+\delta\cdot(\mu+\gamma-R_{S_\infty})}{(1-\delta)\cdot(\Delta-\gamma)+\delta\cdot\Delta}<1. \tag{A7}$$

The threshold $\hat{F}^*_{S_2}$ can be represented as a weighted average of the firm index with zero cashflow in S_2, namely $\bar{F}^*_{S_2}=(\mu-R_{S_2})/(\Delta-\gamma)<1$, and the firm index with zero cashflows in S_∞, namely $\bar{F}_{S_\infty}=(\mu+\gamma-R_{S_\infty})/\Delta<1$.

$$\hat{F}^*_{S_2}=\frac{(1-\delta)\cdot(\Delta-\gamma)}{(1-\delta)\cdot(\Delta-\gamma)+\delta\cdot\Delta}\frac{\mu-R_{S_2}}{\Delta-\gamma}+\frac{\delta\cdot\Delta}{(1-\delta)\cdot(\Delta-\gamma)+\delta\cdot\Delta}\frac{\mu+\gamma-R_{S_\infty}}{\Delta}. \tag{A8}$$

Since our assumptions imply that $\bar{F}^*_{S_2}<\bar{F}_{S_\infty}$, the marginal firm in S_2 requires investment at $t=2$ and has positive private value at $t=\infty$. Naturally, $\hat{F}^*_{S_2}$ is decreasing in both R_{S_2} and R_{S_∞} and is increasing in μ, γ, and δ. Therefore, since $R_{B_2} > R_{G_2}$ and $R_{B_\infty} > R_{G_\infty}$, we have $\hat{F}^*_{B_2} < \hat{F}^*_{G_2}$—i.e., fewer firms has positive private value if the recession is bad at $t = 2$.

Combing these two cases, we have:

$$F^*_{S_2}(F_1) = \min\{F_1, \hat{F}^*_{S_2}\}. \tag{A9}$$

In words, the equilibrium number of firms who survive in S_2 is the lesser of the number of firms who survive at $t = 1$, F_1, and the number of firms with non-negative private value in S_2, $\hat{F}_{S_2}$.

THE INITIAL DATE Finally, we consider what happens at $t = 1$. If the mass of firms that continue operating at $t = 1$ is equal to F_1, then private value of firm f is given by:

$$V_1(f, F_1) = \min\left\{X_1(f, R_1) + (1-\varphi)\delta \cdot \left[(1-p)\cdot V_2(f, R_{G_2}, F^*_{G_2}(F_1)) + p \cdot V_2(f, R_{B_2}, F^*_{B_2}(F_1))\right], 0\right\}, \tag{A10}$$

where $\varphi \in (0,1)$ reflects the credit market frictions that exist at $t = 1$ and $F^*_{S_2}(F_1) = \min\{F_1, \hat{F}^*_{S_2}\}$ is agents' rational expectation of the mass of firms that will continue operating in state S_2 at $t = 2$ if all firms $f \in [0, F_1]$ continue operating at $t = 1$. Thus, the marginal firm who continue operating in the private market equilibrium at $t = 1$ satisfies $0 = V_1(F^*_1, F^*_1)$ or

$$\begin{aligned}
0 = {} & [\mu + \gamma - R_1 - \Delta \times F^*_1] \\
& + (1-\varphi)(1-p)\delta \cdot \left([\mu + \gamma \times F^*_1 - R_{G_2} - \Delta \times F^*_1] + \frac{\delta}{1-\delta}\cdot[\mu + \gamma - R_{G_\infty} - \Delta \times F^*_1]\right)\cdot 1_{\{F^*_1 \le \hat{F}^*_{G_2}\}} \\
& + (1-\varphi)p\delta \cdot \left([\mu + \gamma \times F^*_1 - R_{B_2} - \Delta \times F^*_1] + \frac{\delta}{1-\delta}\cdot[\mu + \gamma - R_{B_\infty} - \Delta \times F^*_1]\right)\cdot 1_{\{F^*_1 \le \hat{F}^*_{B_2}\}}.
\end{aligned} \tag{A11}$$

Given our assumptions, the righthand side of (A11) is decreasing in F^*_1 and the terms on second and third lines on (A11) are always non-negative.

Letting $\bar{F}_1 = (\mu + \gamma - R_1)/\Delta < 1$ denote the index of the firm that generates zero free cashflows at $t = 1$, we assume that $\bar{F}_1 < \hat{F}^*_{G_2}$. This means that there are firms who require outside investment to survive at $t = 1$—i.e., firms with negative free cashflow, that have positive value in state G_2 at $t = 2$.[2] This assumption then implies that the marginal firm who continues operating at $t = 1$ must satisfy $\bar{F}_1 < F^*_1 < \hat{F}^*_{G_2}$.[3]

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[2] If $\hat{F}^*_{G_2} < \bar{F}_1$, it is not worthwhile to invest in any negative free cashflow firms at $t = 1$ and it follows trivially that we have $F^*_1 = \bar{F}_1$.

[3] Specifically, when $\bar{F}_1 < \hat{F}^*_{G_2}$, it cannot be optimal to set $F^*_1 \le \bar{F}_1$ since by continuity firm $f = \bar{F}_1 + \varepsilon$ must then have strictly positive value for some sufficiently small $\varepsilon > 0$. Similarly, it cannot be optimal set $F^*_1 \ge \hat{F}^*_{G_2}$, since then the marginal firm generates strictly negative free cashflow at $t = 1$ and is worthless in both states at $t = 2$. Thus, when $\bar{F}_1 < \hat{F}^*_{G_2}$, we must have $\bar{F}_1 < F^*_1 < \hat{F}^*_{G_2}$.

There are then two relevant cases. In the first case, $\hat{F}^*_{B_2} < F^*_1 < \hat{F}^*_{G_2}$, so the marginal firm who continues operating at $t = 1$ survives in the good state at $t = 2$, but is shut down in the bad state. In this case, the marginal firm that survives at $t = 1$ is given by

$$F_1^* = \frac{(1-\delta)\cdot[\mu+\gamma-R_1]+\ (1-\varphi)(1-p)\delta\cdot[(1-\delta)\cdot(\mu-R_{G_2})+\delta\cdot(\mu+\gamma-R_{G_\infty})]}{(1-\delta)\cdot[\Delta]+\ (1-\varphi)(1-p)\delta\cdot[(1-\delta)\cdot(\Delta-\gamma)+\delta\cdot\Delta]} < 1. \quad \text{(A12)}$$

The expression in (A12) is decreasing in R_1, R_{G_2}, $R_{G\infty}$, p, and φ and is increasing in μ, γ, and δ.

In the second case, $F^*_1 \leq \hat{F}^*_{B_2} < \hat{F}^*_{G_2}$, so the marginal firm who continues operating at $t = 1$ survives in both states at $t = 2$. In this case, the marginal firm that survives at $t = 1$ is given by

$$F_1^* = \frac{(1-\delta)\cdot[\mu+\gamma-R_1]+\ (1-\varphi)\delta\cdot[(1-\delta)\cdot(\mu-\bar{R}_{S_2})+\delta\cdot(\mu+\gamma-\bar{R}_{S_\infty})]}{(1-\delta)\cdot[\Delta]+\ (1-\varphi)\delta\cdot[(1-\delta)\cdot(\Delta-\gamma)+\delta\cdot\Delta]} < 1, \quad \text{(A13)}$$

where $\bar{R}_2 = pR_{B_2} + (1-p)R_{G_2}$ and $\bar{R}_\infty = pR_{B\infty} + (1-p)R_{G\infty}$ are the average recession severities at $t = 2$ and $t = \infty$, respectively. The expression in (A13) is decreasing in R_1, $\bar{R}_2$, $\bar{R}_\infty$, and φ and is increasing in μ, γ, and δ.

Since we must have $\bar{F}_1 < F^*_1$, the marginal firm that is operating at $t = 1$, must fail in state B_2 when $\hat{F}^*_{B_2} < \bar{F}_1$. And, when $\hat{F}^*_{B_2} > \bar{F}_1$, the marginal firm that is operating at $t = 1$ will fail in the bad state when $0 < V_1(\hat{F}^*_{B_2}, \hat{F}^*_{B_2})$, or when

$$\frac{(1-\delta)\cdot(\mu-R_{B_2})+\delta\cdot(\mu+\gamma-R_{B_\infty})}{(1-\delta)\cdot(\Delta-\gamma)+\delta\cdot\Delta} < \frac{(1-\delta)\cdot[\mu+\gamma-R_1]+(1-\varphi)(1-p)\delta\cdot\left((1-\delta)\cdot[\mu-R_{G_2}]+\delta\cdot[\mu+\gamma-R_{G_\infty}]\right)}{(1-\delta)\cdot[\Delta]+(1-\varphi)(1-p)\delta\cdot\left((1-\delta)\cdot[(\Delta-\gamma)]+\delta\cdot[\Delta]\right)}. \quad \text{(A14)}$$

The condition in equation (A14) is easier to satisfy when $R_{B_2}$and $R_{B\infty}$ are larger or when R_1, R_{G_2}, $R_{G\infty}$, p, or φ are smaller.

A.3. The Social Planner's Solution

THE STEADY STATE Again, we start in state S_∞ at $t = \infty$. Since there are no market failures in the long-run, the planner places the same value on firms in the new steady state. Specifically, if firm f survives until $t = \infty$ in state S_∞, its social value is given by the expression for $V_\infty(f, R_{S_2})$ given in equation (A5).

THE INTERIM DATE We again work backward to state S_2 at $t = 2$. We suppose all firms $f \in [0, F_1]$ survived at $t = 1$. If the mass of firms that continue operating in state S_2 at $t = 2$ is equal to F_{S_2}, total social value is given by

$$W_2(R_{S_2},F_1)=\max_{F_{S_2}\le F_1}\left\{\int_0^{F_{S_2}}\left(X_2(f,R_{S_2},F_{S_2})+\delta\cdot V_\infty(f,R_{S_\infty})\right)df\right\}$$
$$=\max_{F_{S_2}\le F_1}\left\{\int_0^{F_{S_2}}\left([\mu+\gamma\times F_{S_2}-R_{S_2}-\Delta\times f]+\frac{\delta}{1-\delta}\cdot[\mu+\gamma-R_{S_\infty}-\Delta\times f]\cdot 1_{\{f\le\bar{F}_{S_\infty}\}}\right)df\right\}. \quad \text{(A15)}$$

Again, there are two cases to consider at S_2. First, some firms that survive at $t=1$ may be shut down by the planner in state S_2 at $t=2$. In this case, $F_{S_2}^{**}=\hat{F}_{S_2}^{**}<F_1$ where $\hat{F}_{S_2}^{**}$ is the unconstrained maximizer of equation (A15). Differentiating the total social value with respect to F_{S_2}, the planner's first order condition for setting $\hat{F}_{S_2}^{**}$ is

$$0=[\mu+\gamma\times F_{S_2}-R_{S_2}-\Delta\times f]+\frac{\delta}{1-\delta}\cdot[\mu+\gamma-R_{S_\infty}-\Delta\times f]\cdot 1_{\{f\le\bar{F}_{S_\infty}\}}+\gamma\times F_{S_2}, \quad \text{(A16)}$$

The first two terms correspond to the private value of the marginal firm and the final term, $\gamma\times F_{S_2}$, reflects the positive spillovers on other firms from keeping this marginal firm alive. We assume that $R_{S_2}-R_{S_\infty}>\gamma$ implying these time-2 spillovers are not so strong that the planner wants to finance firms at $t=2$ that the planner knows will shut down in the steady state. Thus, if firms are shut down by the planner in S_2, the marginal firm $\hat{F}_{S_2}^{**}$ is given by:

$$\hat{F}_{S_2}^{**}=\frac{(1-\delta)\cdot(\mu-R_{S_2})+\delta\cdot(\mu+\gamma-R_{S_\infty})}{(1-\delta)\cdot(\Delta-2\gamma)+\delta\cdot\Delta}. \quad \text{(A17)}$$

Comparing equations (A7) and (A17), we see that $\hat{F}_{S_2}^{**}>\hat{F}_{S_2}^{*}$ when $\gamma>0$. Thus, the planner chooses to keep more firms alive than the private market in both states at $t=2$. This is because, unlike market participants, the planner to internalizes the spillovers that are created by the aggregate demand externality ($\gamma>0$). Alternately, if $\hat{F}_{S_2}^{**}>F_1$, then no firms are shut down in S_2 and the planner sets $F_{S_2}^{**}=F_1$. Thus, combing these two cases, we have:

$$F_{S_2}^{**}(F_1)=\min\{F_1,\hat{F}_{S_2}^{**}\}, \quad \text{(A18)}$$

and the planner's social value function is given by

$$W_2(R_{S_2},F_1)=\int_0^{F_{S_2}^{**}(F_1)}\left([\mu+\gamma\times F_{S_2}^{**}(F_1)-R_{S_2}-\Delta\times f]+\frac{\delta}{1-\delta}\cdot[\mu+\gamma-R_{S_\infty}-\Delta\times f]\cdot 1_{\{f\le\bar{F}_{S_\infty}\}}\right)df. \quad \text{(A19)}$$

THE INITIAL DATE Finally, we consider what happens at $t=1$. If the mass of firms that continue operating at $t=1$ is equal to F_1, then total social value is given by:

$$W_1(F_1)=\int_0^{F_1}X_1(f,R_1)+\delta\cdot\left[(1-p)\cdot W_2(R_{G_2},F_{G_2}^{*}(F_1))+p\cdot W_2(f,R_{B_2},F_{B_2}^{*}(F_1))\right]. \quad \text{(A20)}$$

The first order condition for maximizing equation (A20) is

$$0 = X_1(F_1, R_1) + \delta \cdot \left[(1-p) \cdot \frac{\partial W_2(R_{G_2}, F_{G_2}^{**}(F_1))}{\partial F_{G_2}} \cdot \frac{\partial F_{G_2}^{**}(F_1)}{\partial F_1} + p \cdot \frac{\partial W_2(f, R_{B_2}, F_{B_2}^{**}(F_1))}{\partial F_{B_2}} \cdot \frac{\partial F_{B_2}^{**}(F_1)}{\partial F_1} \right]. \quad \text{(A21)}$$

Writing this out we obtain:

$$\begin{aligned} 0 = {} & [\mu + \gamma - R_1 - \Delta \times F_1^{**}] \\ & + (1-p)\delta \cdot \left([\mu + 2\gamma \times F_1^{**} - R_{G_2} - \Delta \times F_1^{**}] + \frac{\delta}{1-\delta} \cdot [\mu + \gamma - R_{G_\infty} - \Delta \times F_1^{**}] \right) \cdot 1_{\{F_1^{**} \le \hat{F}_{G_2}^{**}\}} \\ & + p\delta \cdot \left([\mu + 2\gamma \times F_1^{*} - R_{B_2} - \Delta \times F_1^{**}] + \frac{\delta}{1-\delta} \cdot [\mu + \gamma - R_{B_\infty} - \Delta \times F_1^{**}] \right) \cdot 1_{\{F_1^{**} \le \hat{F}_{B_2}^{**}\}}. \end{aligned} \quad \text{(A22)}$$

Given our assumptions, the righthand side of (A22) is decreasing in F_1^{**} and the terms on second and third lines in (A22) are always non-negative.

Comparing equation (A22) to (A11), we see how the two underlying market failures, namely (i) the time-1 credit market frictions captured by φ and (ii) the time-2 aggregate demand externality captured by γ, lead the planner's solution (F_1^{**}) to diverge from the private market outcome (F_1^{*}). First, time-1 credit market frictions ($\varphi > 0$), lead the private investors to down weight the benefits of investing to keep firms alive from $t = 1$ to $t = 2$: in effective, credit market frictions lead private investors to behave as if they are more impatient than members of society truly are.[4] Second, when there are aggregate demand externalities ($\gamma > 0$) at $t = 2$, unlike market participants, the planner to internalizes the positive spillovers on other firms that are created by keeping additional firms alive at $t = 2$.[5] Specifically, since the righthand side of (A22) exceeds the righthand side of (A11) for a given F_1 when either $\varphi > 0$ or $\gamma > 0$, we must have $F_1^{**} > F_1^{*}$ when $\varphi > 0$ or $\gamma > 0$.

Since we assumed that $\bar{F}_1 < \hat{F}_{G_2}^{*}$ and since $\hat{F}_{G_2}^{*} < \hat{F}_{G_2}^{**}$, we have $\bar{F}_1 < \hat{F}_{G_2}^{**}$. Thus, the marginal firm that continues at $t = 1$ in the planner's solution satisfies $\bar{F}_1 < F_1^{**} < \hat{F}_{G_2}^{**}$—i.e., the marginal firm requires investment at $t = 1$ and will have strictly positive marginal social value in the good state at $t = 2$. Again, there are two relevant cases. In the first case, $\hat{F}_{B_2}^{**} < F_1^{**} < \hat{F}_{G_2}^{**}$, so the marginal firm at $t = 1$ survives in the good state at $t = 2$, but is shut down by the planner in the bad state. In this case, the marginal firm at $t = 1$ is given by

[4] Formally, when there are credit market frictions at $t = 1$, private investors behave as if the discount factor from $t = 1$ to $t = 2$ is $(1 - \varphi)\delta$, which is less the true discount factor δ. In other words, private market investors effectively use a higher discount rate when discounting the benefits of keeping firms alive from $t = 1$ to $t = 2$.

[5] Formally, this is reflected in the fact that equation (A11) contains a term that is proportional to $\gamma \times F_1^{*}$ whereas equation (A22) contains a term that is proportional to $2\gamma \times F_1^{**}$. Relatedly, since $\hat{F}_{G_2}^{**} > \hat{F}_{G_2}^{*}$ and $\hat{F}_{B_2}^{**} > \hat{F}_{B_2}^{*}$, the planner recognizes at $t = 1$ that the aggregate demand externality will lead her to choose to keep more firms alive in $t = 2$ in both the good and the bad states.

$$F_1^{**} = \frac{(1-\delta)\cdot[\mu+\gamma-R_1] + (1-p)\delta\cdot[(1-\delta)\cdot(\mu-R_{G_2})+\delta\cdot(\mu+\gamma-R_{G_\infty})]}{(1-\delta)\cdot[\Delta] + (1-p)\delta\cdot[(1-\delta)\cdot(\Delta-2\gamma)+\delta\cdot\Delta]}. \quad \text{(A23)}$$

The expression in (A23) is decreasing in R_1, R_{G_2}, R_{G_∞}, and p and is increasing in μ, γ, and δ. It is easy to verify that the expression in (A23) exceeds the corresponding expression in (A12) whenever $\varphi > 0$ or $\gamma > 0$.

In the second case, $F_1^{**} \le \hat{F}_{B_2}^{**} < \hat{F}_{G_2}^{**}$, so the marginal firm at $t = 1$ will survive in both states at $t = 2$. In this case, the marginal firm at $t = 1$ is given by

$$F_1^{**} = \frac{(1-\delta)\cdot[\mu+\gamma-R_1] + \delta\cdot[(1-\delta)\cdot(\mu-\bar{R}_{S_2})+\delta\cdot(\mu+\gamma-\bar{R}_{S_\infty})]}{(1-\delta)\cdot[\Delta] + \delta\cdot[(1-\delta)\cdot(\Delta-2\gamma)+\delta\cdot\Delta]}, \quad \text{(A24)}$$

The expression in (A24) is decreasing in R_1, $\bar{R}_{S_2}$, and $\bar{R}_{S_\infty}$ and is increasing in μ, γ, and δ. Again, the expression in (A24) exceeds the corresponding expression in (A13) whenever $\varphi > 0$ or $\gamma > 0$.

Since we must have $\bar{F}_1 < F_1^{**}$, the marginal firm that is operating at $t = 1$, must fail in state B_2 when $\hat{F}_{B_2}^{**} < \bar{F}_1$. And, when $\hat{F}_{B_2}^{**} > \bar{F}_1$, the marginal firm that is operating at $t = 1$ will fail in the bad state when

$$\frac{(1-\delta)\cdot(\mu-R_{B_2})+\delta\cdot(\mu+\gamma-R_{B_\infty})}{(1-\delta)\cdot(\Delta-2\gamma)+\delta\cdot\Delta} < \frac{(1-\delta)\cdot[\mu+\gamma-R_1] + (1-p)\delta\cdot\left((1-\delta)\cdot[\mu-R_{G_2}]+\delta\cdot[\mu+\gamma-R_{G_\infty}]\right)}{(1-\delta)\cdot[\Delta] + (1-p)\delta\cdot\left((1-\delta)\cdot[(\Delta-2\gamma)]+\delta\cdot[\Delta]\right)}. \quad \text{(A25)}$$

The condition in equation (A25) is easier to satisfy when $R_{B_2}$and R_{B_∞} are larger and when R_1, R_{G_2}, R_{G_∞}, and p are smaller.